Symmetry and Supersymmetry in Nuclear Pairing: Exact Solutions

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Quasi-Spin Algebra

$$\hat{S}_j^+ = \sum_{m>0} (-1)^{(j-m)} a^\dagger_{j\,m} a^\dagger_{j\,-m},$$

$$\hat{S}_j^- = \sum_{m>0} (-1)^{(j-m)} a_{j\,-m} a_{j\,m}$$

$$\hat{S}_j^0 = \frac{1}{2} \sum_{m>0} \left(a^\dagger_{j\,m} a_{j\,m} + a^\dagger_{j\,-m} a_{j\,-m} - 1, \right)$$

$$\hat{S}_j^0 = \hat{N}_j - \frac{1}{2}\Omega_j.$$

$\Omega_j = j + \frac{1}{2}$ = the maximum number of pairs that can occupy the level j

$$\hat{N}_j = \frac{1}{2} \sum_{m>0} \left(a^\dagger_{j\,m} a_{j\,m} + a^\dagger_{j\,-m} a_{j\,-m} \right).$$

Exactly solvable cases:

- Quasi-spin limit (Kerman)

$$\hat{H} = -|G| \sum_{jj'} \hat{S}_j^+ \hat{S}_{j'}^- .$$

- Richardson's solution:

$$\hat{H} = \sum_{jm} \epsilon_j a_{j\,m}^\dagger a_{j\,m} - |G| \sum_{jj'} \hat{S}_j^+ \hat{S}_{j'}^- .$$

- Gaudin's model - closely related to Richardson's.
- The limit in which the energy levels are degenerate (the first term is a constant for a given number of pairs):

$$\hat{H} = -|G| \sum_{jj'} c_j^* c_{j'} \hat{S}_j^+ \hat{S}_{j'}^- .$$

  (Draayer, Pan, Balantekin, Pehlivan, de Jesus)
- Most general separable case with two shells (Balantekin and Pehlivan).

Degenerate Solution

Define

$$\hat{S}^+(0) = \sum_j c_j^* \hat{S}_j^+ \quad \text{and} \quad \hat{S}^-(0) = \sum_j c_j \hat{S}_j^-,$$

$$\hat{H} = -|G|\hat{S}^+(0)\hat{S}^-(0).$$

In the 1970's Talmi showed that under certain assumptions, a state of the form

$$\hat{S}^+(0)|0\rangle = \sum_j c_j^* \hat{S}_j^+ |0\rangle, \quad |0\rangle\text{: particle vacuum}$$

is an eigenstate of a class of Hamiltonians including the one above. Indeed

$$\hat{H}\hat{S}^+(0)|0\rangle = \left(-|G|\sum_j \Omega_j |c_j|^2\right)\hat{S}^+(0)|0\rangle$$

What about other one-pair states?

For example for two levels j_1 and j_2, the orthogonal state

$$\left(\frac{c_{j_2}}{\Omega_{j_1}} \hat{S}^{+}_{j_1} - \frac{c_{j_1}}{\Omega_{j_2}} \hat{S}^{+}_{j_2} \right) |0\rangle,$$

is also an eigenstate with E=0.

Energy/$(-\|G\|)$	**State**
0	$\left(-\frac{c_{j_2}}{\Omega_{j_1}} \hat{S}^{+}_{j_1} + \frac{c_{j_1}}{\Omega_{j_2}} \hat{S}^{+}_{j_2} \right) \|0\rangle$
$\Omega_{j_1} \|c_{j_1}\|^2 + \Omega_{j_1} \|c_{j_1}\|^2$	$\left(c^{*}_{j_1} \hat{S}^{+}_{j_1} + c^{*}_{j_2} \hat{S}^{+}_{j_2} \right) \|0\rangle$

States with N=1 for two shells

Define

$$\hat{S}^+(x) = \sum_j \frac{c_j^*}{1 - |c_j|^2 x} \hat{S}_j^+ \quad \text{and} \quad \hat{S}^-(x) = \sum_j \frac{c_j}{1 - |c_j|^2 x} \hat{S}_j^- .$$

Then eigenstates are of the form

$$\hat{S}^+(x)\hat{S}^+(y)\cdots\hat{S}^+(z)|0\rangle$$

F. Pan, J.P. Draayer, W.E. Ormand, Phys. Lett. B **422**, 1 (1998)

$$\hat{K}^0(x) = \sum_j \frac{1}{1/|c_j|^2 - x} \hat{S}_j^0$$

$$[\hat{S}^+(x), \hat{S}^-(0)] = [\hat{S}^+(0), \hat{S}^-(x)] = 2K^0(x)$$

$$[\hat{K}^0(x), \hat{S}^\pm(y)] = \pm \frac{\hat{S}^\pm(x) - \hat{S}^\pm(y)}{x - y}$$

$$\hat{H} = -|G| \sum_{jj'} c_j^* c_{j'} \hat{S}_j^+ \hat{S}_{j'}^-$$

$$\hat{S}^+(x) = \sum_j \frac{c_j^*}{1 - |c_j|^2 x} \hat{S}_j^+$$

❶

$$\hat{S}^+(0)\hat{S}^+(z_1^{(N)}) \ldots \hat{S}^+(z_{N-1}^{(N)})|0\rangle$$

is an eigenstate if the following Bethe ansatz equations are satisfied:

$$\sum_j \frac{-\Omega_j/2}{1/|c_j|^2 - z_m^{(N)}} = \frac{1}{z_m^{(N)}} + \sum_{k=1(k\neq m)}^{N-1} \frac{1}{z_m^{(N)} - z_k^{(N)}} \qquad m = 1, 2, \ldots N-1.$$

$$E_N = -|G| \left(\sum_j \Omega_j |c_j|^2 - \sum_{k=1}^{N-1} \frac{2}{z_k^{(N)}} \right)$$

❷

$$\hat{S}^+(x_1^{(N)})\hat{S}^+(x_2^{(N)}) \ldots \hat{S}^+(x_N^{(N)})|0\rangle$$

is an eigenstate with zero energy if the following Bethe ansatz equations are satisfied:

$$\sum_j \frac{-\Omega_j/2}{1/|c_j|^2 - x_m^{(N)}} = \sum_{k=1(k\neq m)}^{N} \frac{1}{x_m^{(N)} - x_k^{(N)}} \qquad \text{for every} \quad m = 1, 2, \ldots, N$$

These are eigenstates if the shell is at most half full!

What if the available states are more than half full? There are degeneracies:

No. of Pairs	**Energy/$(-\|G\|)$**	**State**
1	$\sum_j \Omega_j \|c_j\|^2$	$\hat{S}^+(0)\|0\rangle$
N_{max}	$\sum_j \Omega_j \|c_j\|^2$	$\|\bar{0}\rangle$

$|0\rangle$: particle vacuum

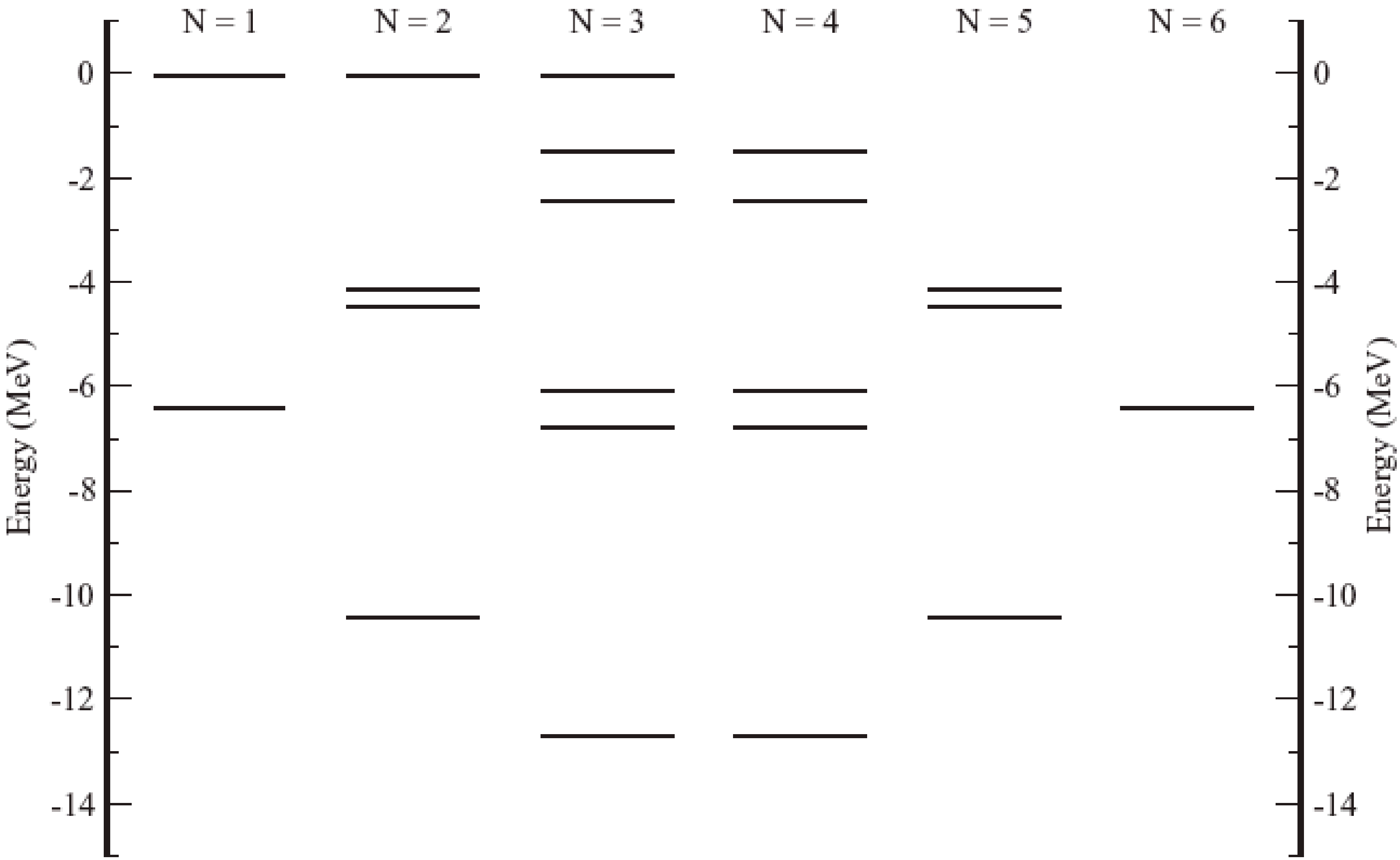
$|\bar{0}\rangle$: state where all levels are completely filled

No. of Pairs	State
N	$\hat{S}^{+}(0)\hat{S}^{+}(z_1^{(N)})\ldots\hat{S}^{+}(z_{N-1}^{(N)})\vert 0\rangle$
$N_{max}+1-N$	$\hat{S}^{-}(z_1^{(N)})\hat{S}^{-}(z_2^{(N)})\ldots\hat{S}^{-}(z_{N-1}^{(N)})\vert\bar{0}\rangle$

$$E = -G\left(\sum_j \Omega_j |c_j|^2 - \sum_{k=1}^{N-1}\frac{2}{z_k^{(N)}}\right)$$

$$\sum_j \frac{-\Omega_j/2}{1/|c_j|^2 - z_m^{(N)}} = \frac{1}{z_m^{(N)}} + \sum_{k=1(k\neq m)}^{N-1}\frac{1}{z_m^{(N)} - z_k^{(N)}}$$

A.B. Balantekin, J. de Jesus, and Y. Pehlivan, Phys. Rev. C **75**, 064304 (2007)



sd-shell with $0d_{5/2}$,$0d_{3/2}$, and $1s_{1/2}$

$$\hat{H}_{SC} \sim -|G|\hat{S}^{+}(0)\hat{S}^{-}(0),$$

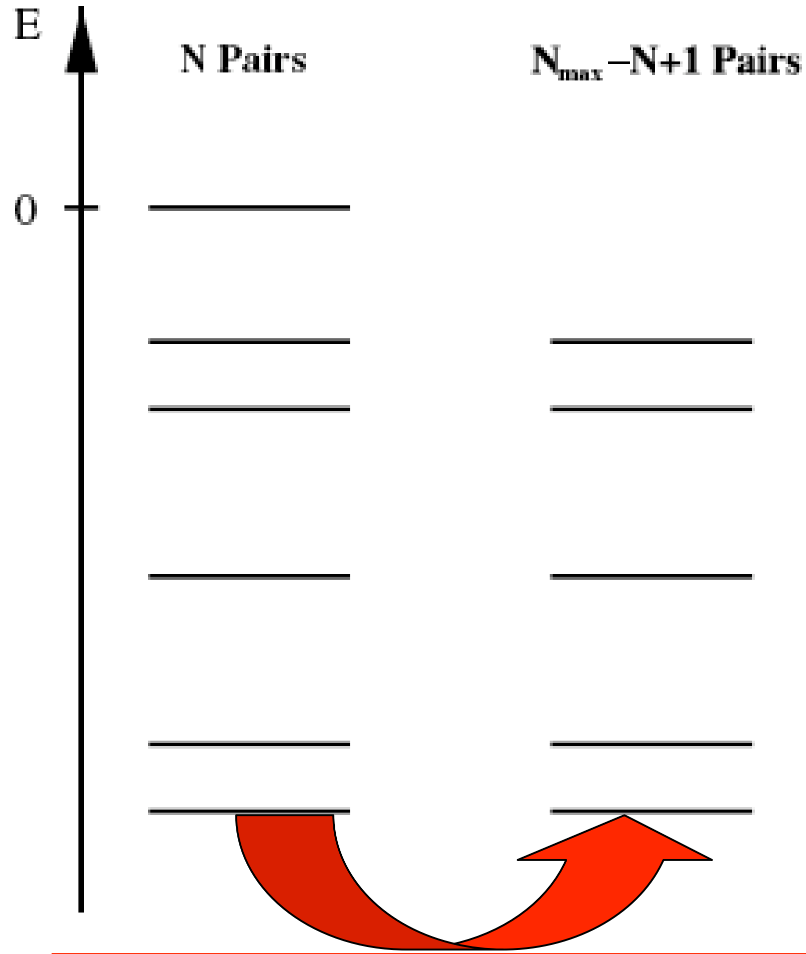
$$\hat{S}^{+}(0) = \sum_j c_j^* \hat{S}_j^{+} \quad \text{and} \quad \hat{S}^{-}(0) = \sum_j c_j \hat{S}_j^{-}.$$

$$\hat{T} = \exp\left(-i\frac{\pi}{2}\sum_i (\hat{S}_i^{+} + \hat{S}_i^{-})\right) \quad \Longrightarrow \quad \hat{T}|0\rangle = |\bar{0}\rangle$$

$$\hat{B}^{-} = \hat{T}^{\dagger}\hat{S}^{-}(0), \qquad \hat{B}^{+} = \hat{S}^{+}(0)\hat{T}.$$

- SUSY QM tells us the partner Hamiltonians B^+B^- and B^-B^+ have the same energy spectra except the extremal (usually ground) state.
- In this case the partner Hamiltonians happens to be identical, so this supersymmetry connects the particle and hole states.

SUSY



$$\hat{T} = \exp\left(-i\frac{\pi}{2}\sum_i(\hat{S}_i^+ + \hat{S}_i^-)\right)$$

A.B. Balantekin and Y. Pehlivan, J. Phys. G **34**, 1783 (2007).

Exact solution with two non-degenerate levels j_1 and j_2

$$\frac{\hat{H}}{|G|} = \sum_j 2\varepsilon_j \hat{S}_j^0 - \sum_{jj'} c_j^* c_{j'} \hat{S}_j^+ \hat{S}_{j'}^- + \sum_j \varepsilon_j \Omega_j$$

$$J^+(x) = \sum_j \frac{c_j^*}{2\varepsilon_j - |c_j|^2 x} S_j^+$$

$$J^+(x_1) J^+(x_2) \dots J^+(x_N) |0\rangle$$

$$E_N = -\sum_{n=1}^{N} \frac{\delta x_n}{\beta - x_n}$$

Bethe ansatz

$$\sum_j \frac{\Omega_j |c_j|^2}{2\varepsilon_j - |c_j|^2 x_k} = \frac{\beta}{\beta - x_k} + \sum_{n=1(\neq k)}^{N} \frac{2}{x_n - x_k}$$

$$\beta = 2\frac{\varepsilon_{j_1} - \varepsilon_{j_2}}{|c_{j_1}|^2 - |c_{j_2}|^2}$$

$$\delta = 2\frac{\varepsilon_{j_2}|c_{j_1}|^2 - \varepsilon_{j_1}|c_{j_2}|^2}{|c_{j_1}|^2 - |c_{j_2}|^2}$$

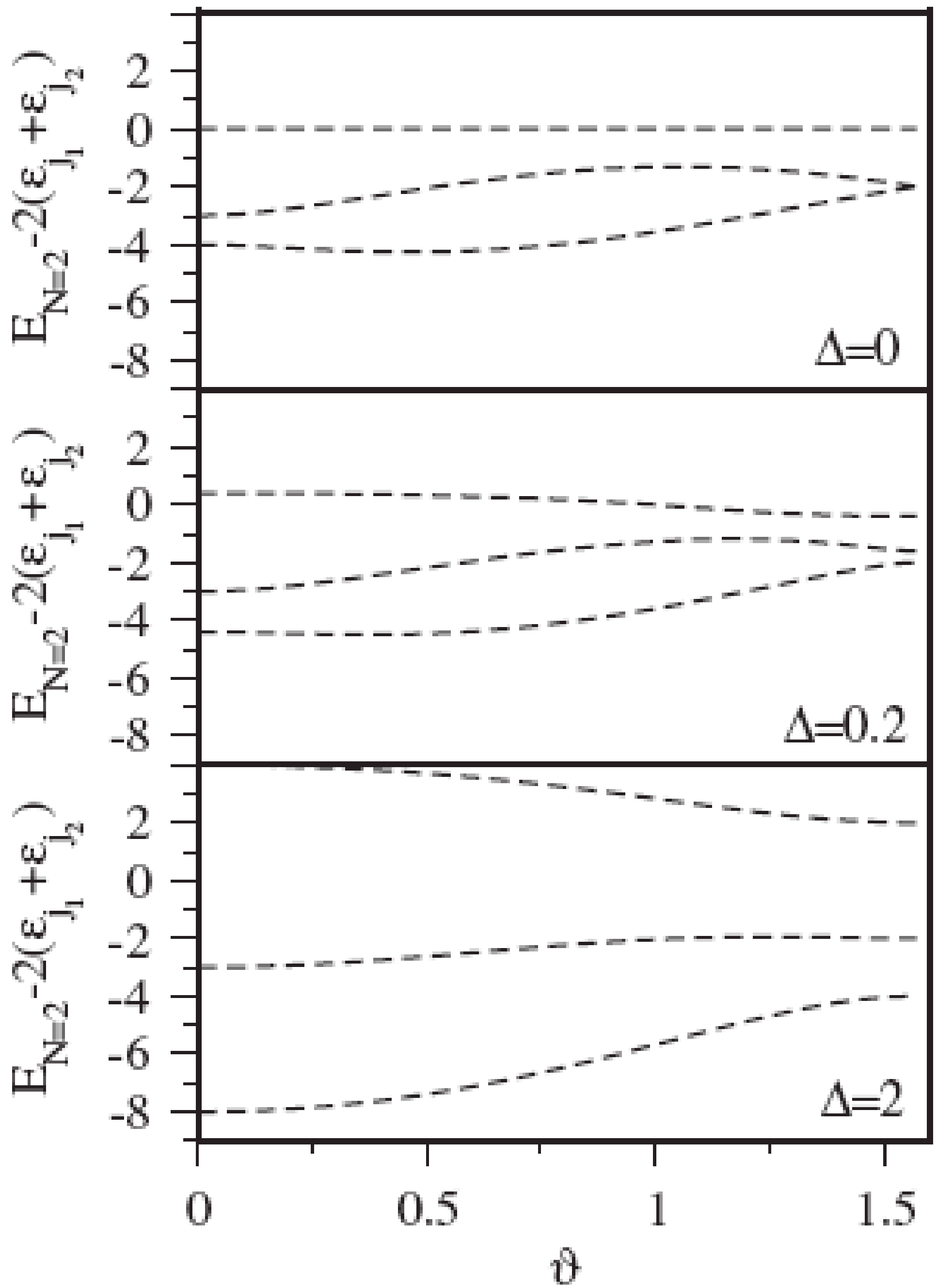
Balantekin and Pehlivan, Phys. Rev. C **76**, 051001(R) (2007).

Exact solution with j_1=3/2 and j_2=5/2

$\cos\vartheta = c_1$

$\sin\vartheta = c_2$

$\Delta = \epsilon_1 - \epsilon_2$



What is next?

- We have an exact solution of the s-wave pairing problem with two non-degenerate orbitals in terms of the solutions of the Bethe ansatz equations. This problem is of course diagonalizable in an SU(2) x SU(2) basis.
- It seems to be very difficult to generalize the Bethe ansatz method to the case of three non-degenerate orbitals. (Of course this problem is also diagonalizable in an SU(2) X SU(2) X SU(2) basis).
- The Bethe ansatz method is generalizable to at least some d-wave pairing situations.

Talmi's conditions for exact solutions:

$$H|0\rangle = 0$$

Hamiltonian with at most 2-body interactions

Doubly-magic core

$$[[H, S^\dagger], S^\dagger] = W\left(S^\dagger\right)^2 \quad \Rightarrow \quad HS^\dagger|0\rangle \propto S^\dagger|0\rangle$$

$$D_M^\dagger = \sum_{jj'} \alpha_{jj'} \left(a_j^\dagger \times a_{j'}^\dagger\right)_M^{(2)}$$

$$[[H, S^\dagger], D_M^\dagger] = WS^\dagger D_M^\dagger \quad \Rightarrow \quad HD_M^\dagger|0\rangle \propto D_M^\dagger|0\rangle$$

Note the suggestion of an algebraic structure. In fact, Ginocchio model satisfies these double commutators. Can we generalize the Bethe ansatz method if we have an algebraic framework?

The answer is YES!

$$[J^+(\lambda), J^-(\mu)] = 2\frac{J^0(\lambda) - J^0(\mu)}{\lambda - \mu},$$

$$[J^0(\lambda), J^{\pm}(\mu)] = \pm\frac{J^{\pm}(\lambda) - J^{\pm}(\mu)}{\lambda - \mu},$$

$$[J^0(\lambda), J^0(\mu)] = [J^{\pm}(\lambda), J^{\pm}(\mu)] = 0$$

Gaudin algebra

$J^-(\lambda)|0\rangle = 0$, $J^0(\lambda)|0\rangle = W(\lambda)|0\rangle$

$$H(\lambda) = J^0(\lambda)J^0(\lambda) + \frac{1}{2}J^+(\lambda)J^-(\lambda) + \frac{1}{2}J^-(\lambda)J^+(\lambda)$$

$$[H(\lambda), H(\mu)] = 0$$

$$|\xi_1, \xi_2, \ldots, \xi_n > \equiv J^+(\xi_1)J^+(\xi_2)\ldots J^+(\xi_n)|0>$$

$$E_n(\lambda) = \left[W(\lambda)^2 - W'(\lambda)\right] - 2\sum_{\alpha=1}^{n}\frac{W(\lambda) - W(\xi_\alpha)}{\lambda - \xi_\alpha}.$$

$$W(\xi_\alpha) = \sum_{\substack{\beta=1 \\ (\beta\neq\alpha)}}^{n}\frac{1}{\xi_\alpha - \xi_\beta} \quad \text{for} \quad \alpha = 1, 2, \ldots, n.$$

The answer is YES!

Gaudin algebra

$$[J^+(\lambda), J^-(\mu)] = 2\frac{J^0(\lambda) - J^0(\mu)}{\lambda - \mu},$$

$$[J^0(\lambda), J^\pm(\mu)] = \pm\frac{J^\pm(\lambda) - J^\pm(\mu)}{\lambda - \mu},$$

$$[J^0(\lambda), J^0(\mu)] = [J^\pm(\lambda), J^\pm(\mu)] = 0$$

$$J^-(\lambda)|0\rangle = 0, \quad J^0(\lambda)|0\rangle = W(\lambda)|0\rangle$$

$$H(\lambda) = J^0(\lambda)J^0(\lambda) + \frac{1}{2}J^+(\lambda)J^-(\lambda) + \frac{1}{2}J^-(\lambda)J^+(\lambda)$$

$$[H(\lambda), H(\mu)] = 0$$

$$J^0(\lambda) = \sum_{i=1}^{N} \frac{\hat{S}_i^0}{\epsilon_i - \lambda}$$

$$J^\pm(\lambda) = \sum_{i=1}^{N} \frac{\hat{S}_i^\pm}{\epsilon_i - \lambda}.$$

The answer is YES!

Gaudin algebra

$$[J^+(\lambda), J^-(\mu)] = 2\frac{J^0(\lambda) - J^0(\mu)}{\lambda - \mu},$$

$$[J^0(\lambda), J^\pm(\mu)] = \pm\frac{J^\pm(\lambda) - J^\pm(\mu)}{\lambda - \mu},$$

$$[J^0(\lambda), J^0(\mu)] = [J^\pm(\lambda), J^\pm(\mu)] = 0$$

$$J^-(\lambda)|0\rangle = 0 \qquad J^0(\lambda)|0\rangle = W(\lambda)|0\rangle$$

$$H(\lambda) = J^0(\lambda)J^0(\lambda) + \frac{1}{2}J^+(\lambda)J^-(\lambda) + \frac{1}{2}J^-(\lambda)J^+(\lambda) \qquad [H(\lambda), H(\mu)] = 0$$

$$\lim_{\lambda \to \epsilon_k} (\lambda - \epsilon_k)\,(H(\lambda) + 2\mathbf{c}\cdot\mathbf{S}) = R_k$$

$$\sum_i R_i = -2\mathbf{c}\cdot\sum_k \mathbf{S}_k \qquad \sum_i \epsilon_i R_i = -2\sum_i \epsilon_i \mathbf{c}\cdot\mathbf{S}_i - 2\sum_{i\neq j}\mathbf{S}_i\cdot\mathbf{S}_j$$

$$\mathbf{c} = (0, 0, -1/2|G|)$$

$$\frac{H}{|G|} = \sum_i \epsilon_i R_i + |G|^2\Big(\sum_i R_i\Big)^2 - |G|\sum_i R_i + \cdots$$

The answer is YES!

$$[J^+(\lambda), J^-(\mu)] = 2\frac{J^0(\lambda) - J^0(\mu)}{\lambda - \mu},$$

$$[J^0(\lambda), J^\pm(\mu)] = \pm\frac{J^\pm(\lambda) - J^\pm(\mu)}{\lambda - \mu},$$

$$[J^0(\lambda), J^0(\mu)] = [J^\pm(\lambda), J^\pm(\mu)] = 0$$

Gaudin algebra

$J^-(\lambda)|0\rangle = 0$, $J^0(\lambda)|0\rangle = W(\lambda)|0\rangle$

$$H(\lambda) = J^0(\lambda)J^0(\lambda) + \frac{1}{2}J^+(\lambda)J^-(\lambda) + \frac{1}{2}J^-(\lambda)J^+(\lambda)$$

$$[H(\lambda), H(\mu)] = 0$$

$$J^0(\lambda) = \sum_{i=1}^{N} \frac{\hat{S}_i^0}{\epsilon_i - \lambda}$$

$$J^\pm(\lambda) = \sum_{i=1}^{N} \frac{\hat{S}_i^\pm}{\epsilon_i - \lambda}.$$

Change these with the generators of the new algebra

The answer is YES!

Find the new infinite algebra

Write the new H in analogy with the Casimir

$$J^0(\lambda) = \sum_{i=1}^{N} \frac{\hat{S}_i^0}{\epsilon_i - \lambda} \qquad J^{\pm}(\lambda) = \sum_{i=1}^{N} \frac{\hat{S}_i^{\pm}}{\epsilon_i - \lambda}.$$

Change these with the generators of the new algebra

Thank you!

Grazie!