Paths to understanding nuclear structure

Observe the phenomena

Develop empirical formulae

Develop phenomenological models

Give models a microscopic interpretation

Express as submodels of the shell model

Define shell model coupling schemes and effective interactions

Make observations on many nuclei.
Plot results in many ways.
Observe patterns.

See if the models support or conflict with the many-nucleon shell model

Learn what the phenomena have to say about the many-nucleon shell model?

The subject of this talk

What does the shell model have to say about deformation and rotations?

What do deformations and rotations have to say about the shell model?

Collective phenomena have been well studied and explained in terms of the Bohr model and more sophisticated collective models. In this talk, I discuss their implication for a theory of effective interactions in deformed nuclei.

The Bohr model

is a liquid drop model with a shape defined by its surface.

It has 5 quadrupole shape coordinates

$$R(\theta,\varphi) = R_0[1+\alpha_\nu^* Y_{2\nu}(\theta,\varphi)+\cdots]$$

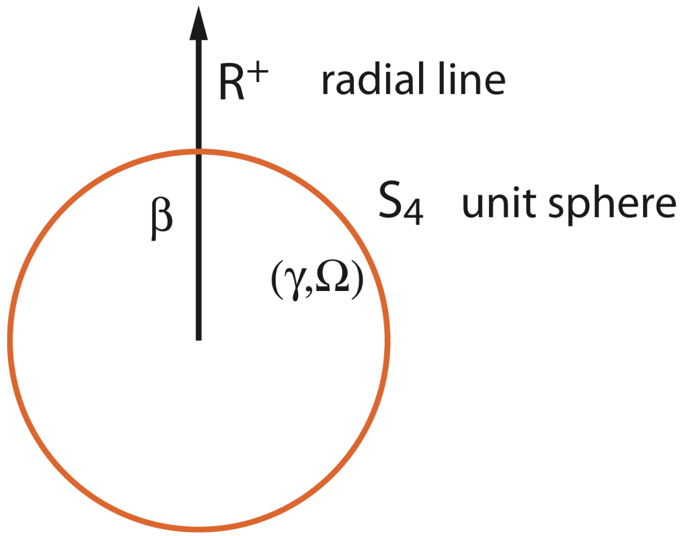
and a variety of Hamiltonians of the form

$$\hat{H} = \frac{1}{2B}\sum_\nu \hat{\pi}^\nu \hat{\pi} + V(\alpha)$$

$$\hat{\pi}^\nu = -i\hbar\frac{\partial}{\partial\alpha_\nu}, \quad \hat{\pi}_\nu = -i\hbar\frac{\partial}{\partial\alpha_\nu^*}$$

It also has very rich geometric and algebraic structures.

Geometric structure



The configuration space is a product manifold

$$\mathbb{R}^{+} \quad \times \quad \mathrm{S}_4$$

$$\Psi_{\lambda\mu;\nu LM} = \frac{1}{\beta^2}\mathcal{R}^{\lambda}_{\mu}(\beta)\,\mathcal{Y}_{\nu LM}(\gamma,\Omega)$$

Algebraic structure

$$\mathrm{SU}(1,1) \quad \times \quad \mathrm{SO}(5) \supset \mathrm{SO}(3)$$

$$\mathcal{R}^{\lambda}_{\mu}(\beta) \quad \times \quad \mathcal{Y}_{\nu LM}(\gamma,\Omega)$$

From these structures we derive basis wave functions as products of radial wave functions and SO(5) spherical harmonics. We also obtain SO(5) Clebsch-Gordan coefficients, and matrix elements of all model observables of interest.

Successes and limitations of the Bohr model

Successes

- The Bohr model provides a qualitative description of nuclear rotational bands.
- It provides the language for the description of nuclear rotations and vibrations.
- It has spawned many more detailed and more successful collective models.

Limitations

- It does not explain the large variety of rotational bands for which extra intrinsic degrees of freedom would appear to be required.
- Observed inertial parameters do not have the irrotational-flow values of a quantum fluid

The many-nucleon collective model

- **The fundamental defect of the Bohr model is that its shape coordinates do not have a microscopic interpretation**

replace with quadrupole moments $$Q_{ij} = \sum_{n=1}^{A} x_{ni} x_{nj}$$

corresponding velocities are then $$\dot{Q}_{ij} = \sum_{n=1}^{A} \left(\dot{x}_{ni} x_{nj} + x_{ni} \dot{x}_{nj} \right)$$

Thus, we define $$\hat{P}_{ij} = \sum_{n=1}^{A} \left(\hat{p}_{ni} x_{nj} + x_{ni} \hat{p}_{nj} \right)$$

- **This makes a huge difference; e.g. its quantisation is given by**

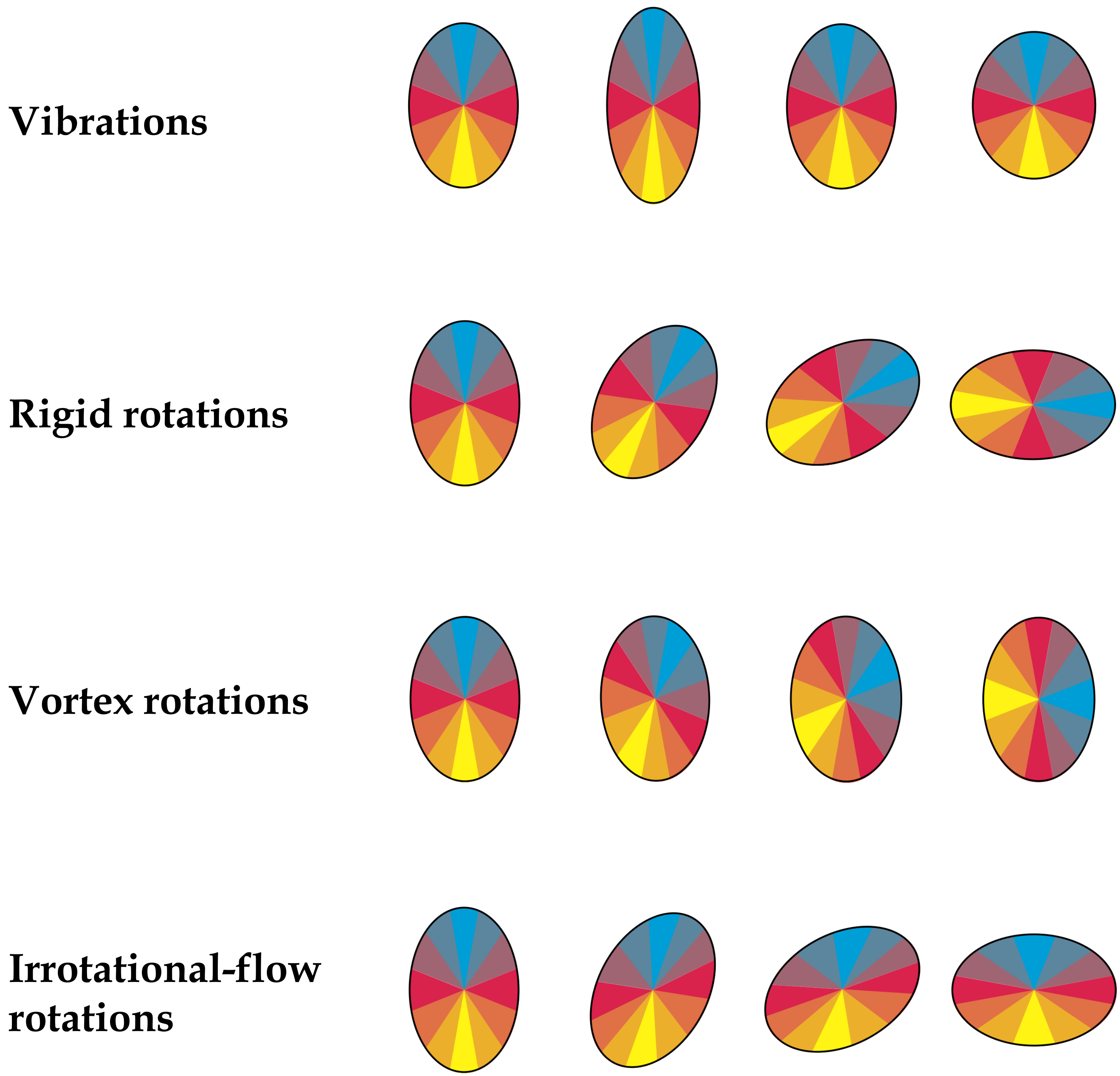
$$\hat{Q}_{ij} \Psi(x) = \sum_{n=1}^{A} x_{ni} x_{nj} \Psi(x), \qquad \hat{P}_{ij} \Psi(x) = \sum_{n=1}^{A} \left(x_{ni} \hat{p}_{nj} + \hat{p}_{ni} x_{nj} \right) \Psi(x)$$

where $$\hat{p}_{nj} = -\partial / \partial x_{nj}$$

and new commutation relations are obtained $$\left[\hat{Q}_{ij}, \hat{P}_{kl} \right] = i\hbar (\delta_{il} \hat{Q}_{kj} + \cdots)$$

The new Lie algebra has irreps with different vorticities

$$\mathcal{L} = 0,\ 1,\ 2,\ 3,\ \cdots$$


Vibrations
Rigid rotations
Vortex rotations
Irrotational-flow rotations

Embedding the microscopic collective model in the shell model

We next want to construct representations on subspaces of the shell model.

This becomes easy if the model is augmented to include the moments of momentum in its Lie algebra

$$K_{ij} = \sum_{n=1}^{A} p_{ni} p_{nj}$$

The augmented microscopic collective model is then the symplectic model with Lie algebra

$$\{Q_{ij}, P_{ij}, K_{ij}\}$$

This Lie algebra contains the many-nucleon kinetic energy, the spherical and deformed harmonic oscillators, and much else

$$T = \frac{1}{2m} \sum_{ni} p_{ni}^2 \qquad H = \frac{1}{2M} \sum_{ni} p_{ni}^2 + \frac{1}{2} M\omega^2 \sum_{ni} x_{ni}^2$$

Its dynamical group contains the SU(3) group as a subgroup

$$\mathrm{Sp}(3,R) \supset \mathrm{U}(3) \supset \mathrm{U}(1) \times \mathrm{SU}(3) \supset \mathrm{SO}(3)$$

Irreps of $\mathrm{Sp}(3,R) \supset \mathrm{SU}(3)$

An irrep can be constructed starting from any lowest-grade SU(3) irrep and applying monopole/quadrupole raising operators.

$$\hat{Q}_{ij} = \hat{Q}_{ij}^{(0)} + \hat{Q}_{ij}^{(+2)} + \hat{Q}_{ij}^{(-2)}$$

$\hat{Q}_{ij}^{(0)}$ Elliott's quadrupole operators

$\hat{Q}_{ij}^{(+2)}$ one-phonon giant resonance raising operators

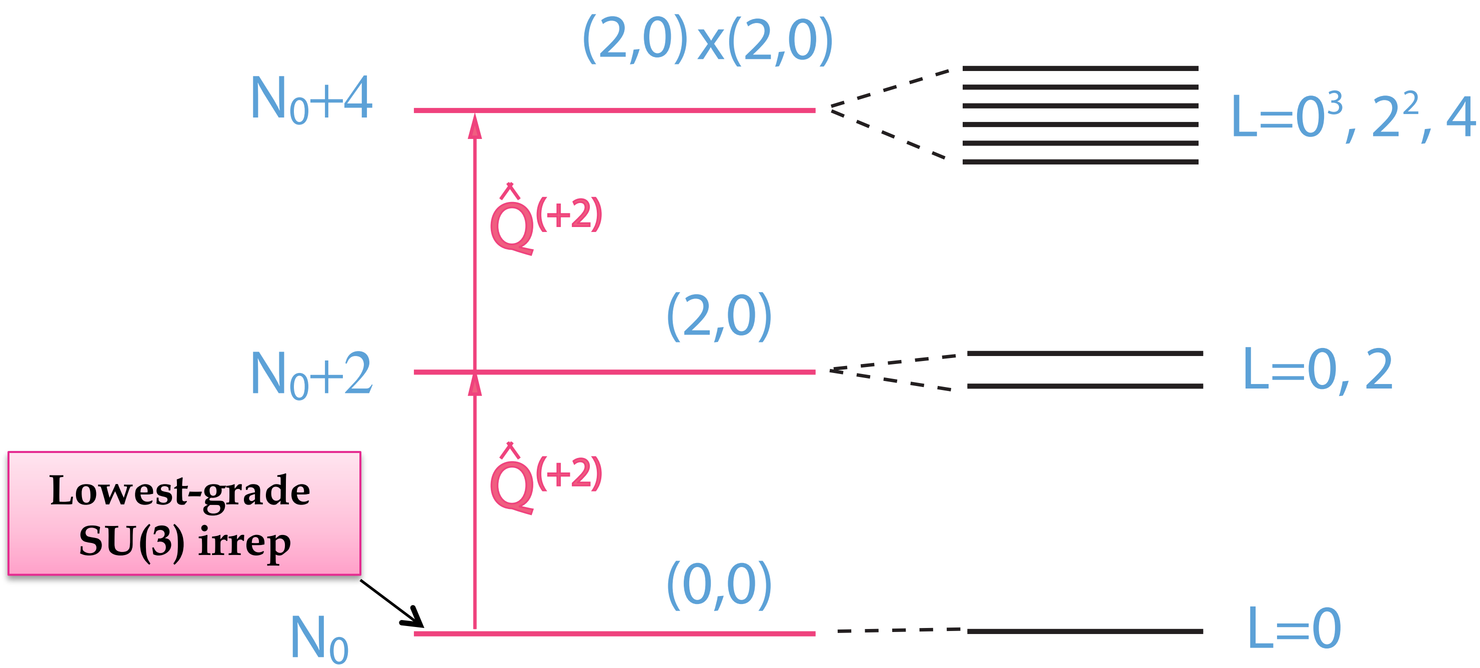
$\hat{Q}_{ij}^{(-2)}$ one-phonon giant resonance lowering operators

Zero-phonon giant monopole/quadrupole states are said to be lowest-grade states. Thus, we can start with a lowest-grade SU(3) irrep and build an Sp(3,R) irrep upon it by repeated application to it of the giant-resonance raising operators.

There are three kinds of Sp(3,R) irrep;
spherical, axially symmetric, and triaxial.

Basis states for a spherical representation

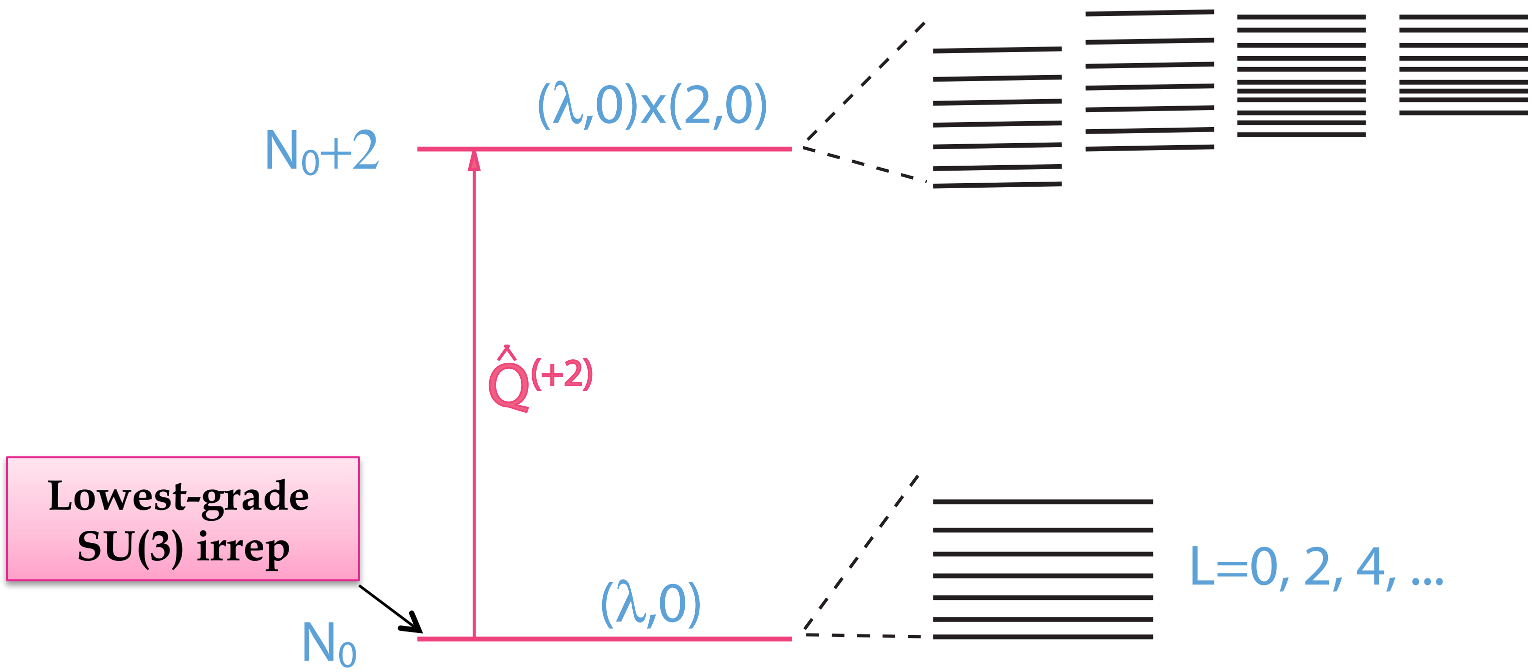
$N(\lambda,\mu) = N_0(0,0)$ irrep



There is a small mixing of N=0 and N>0 irreps giving rise to vibrational ground state correlations.

Basis states for an axially symmetric representation

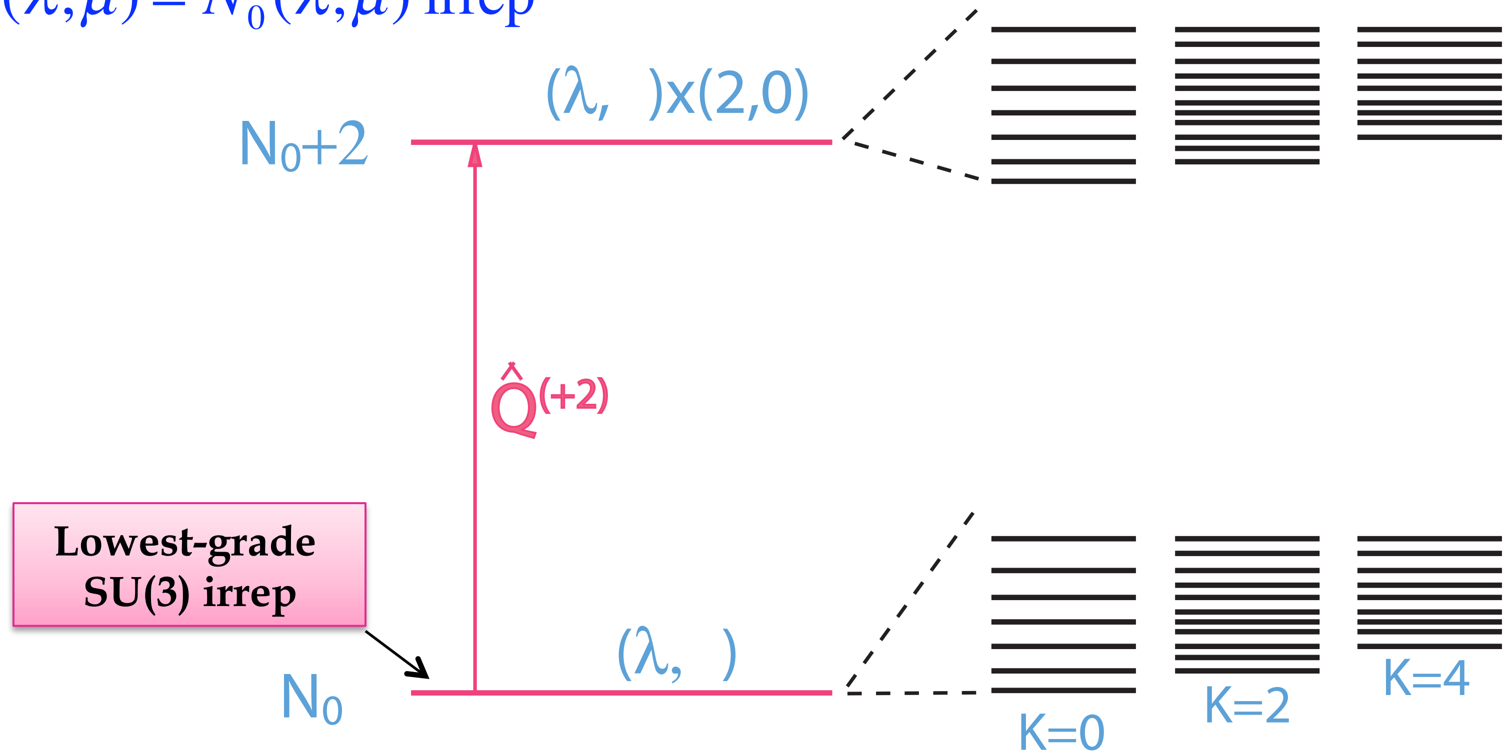
$N(\lambda,\mu) = N_0(\lambda,0)$ or $N_0(0,\mu)$ irrep



There is now a strong mixing of N=0 and N>0 irreps giving rise to a renormalised ground state rotational band.

Basis states for a triaxial representation

$N(\lambda,\mu) = N_0(\lambda,\mu)$ irrep



There is now a strong mixing of N=0 and N>0 irreps giving rise to renormalised ground state rotational bands.

Shell-model description of the low-energy states of rotational nuclei

The shell-model space is a direct sum of irreducible representations of

$$\underbrace{Sp(3,R) \times U(4)}_{\text{Wigner supermultiplet group}} \supset \underbrace{U(3)}_{\text{space}} \times \underbrace{SU(2)_S}_{\text{spin}} \times \underbrace{SU(2)_T}_{\text{isospin}}$$

Ideally we would diagonalise a Hamiltonian with realistic interactions in a space of relevant Sp(3,R) x U(4) representations.

Challenges:

- Diagonalisation in a single Sp(3,R) irrep is difficult -- but possible.
- We need to determine a finite number of most appropriate representations.
- A diagonalisation in such a space requires a new theory of effective interactions

First thoughts on an effective shell model for deformed nuclei

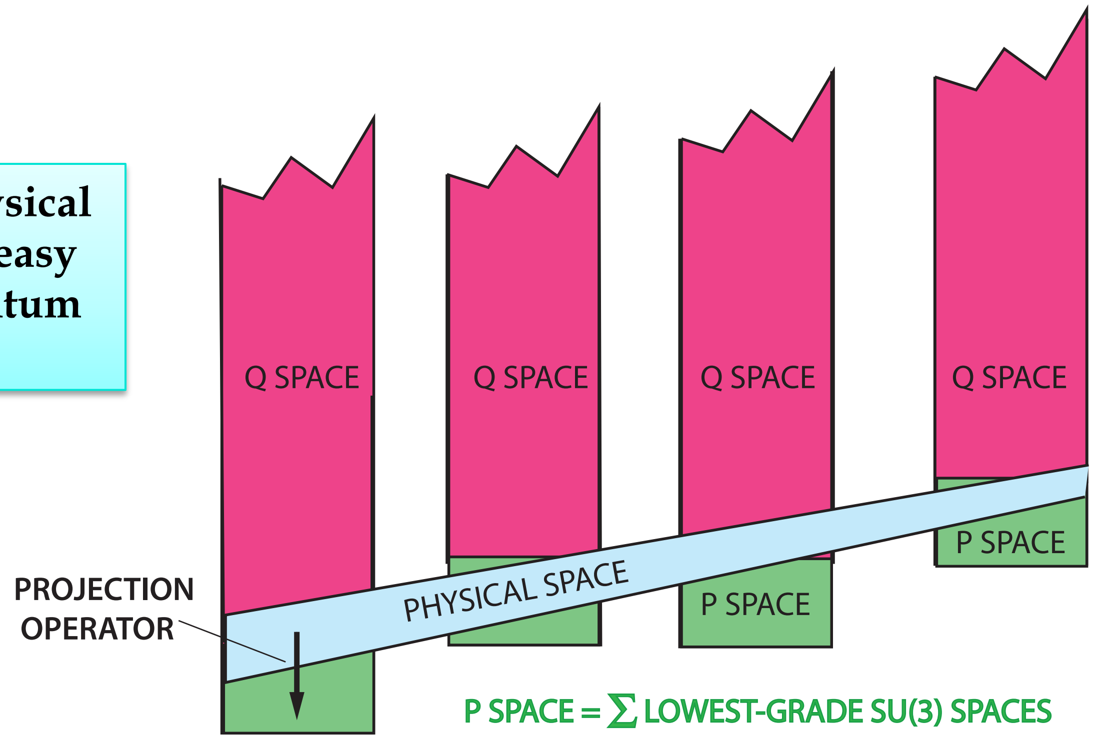
1. **Select an appropriate subset of Sp(3,R) representations;**

   each irrep is characterised by a lowest-grade U(3) subspace with the same quantum numbers (see next slide)

2. **Define the active model P space as a direct sum of these lowest-grade subspaces**

Projection from the physical space to the P space is easy and preserves the quantum numbers

$(\lambda\mu)\; L\; S\; T\;\; J^{\pi}$



Selection of relevant Sp(3,R) subspaces

The experimental choice:

- **The SU(3) model gives a map**

$$(\lambda\mu) \quad \rightarrow \quad \text{E2 rates and quadrupole moments}$$

- **The Sp(3.R) model (with self-consistency) gives a map**

$$(\lambda\mu) \quad \rightarrow \quad \mathrm{e}_{eff} \times \text{SU(3) E2 rates and quadrupole moments}$$

- **Conversely, experimental observations gives a map**

$$\text{E2 rates and quadrupole moments} \quad \rightarrow \quad (\lambda\mu)$$

Example:

$$^{168}\mathrm{Er} \qquad \langle\lambda\rangle \sim 88, \quad \langle\mu\rangle \sim 11$$

- **consistent with the Nilsson model for the observed deformation which also gives**

$$N = N_0 + 16$$

where N_0 is the minimum spherical harmonic oscillator value

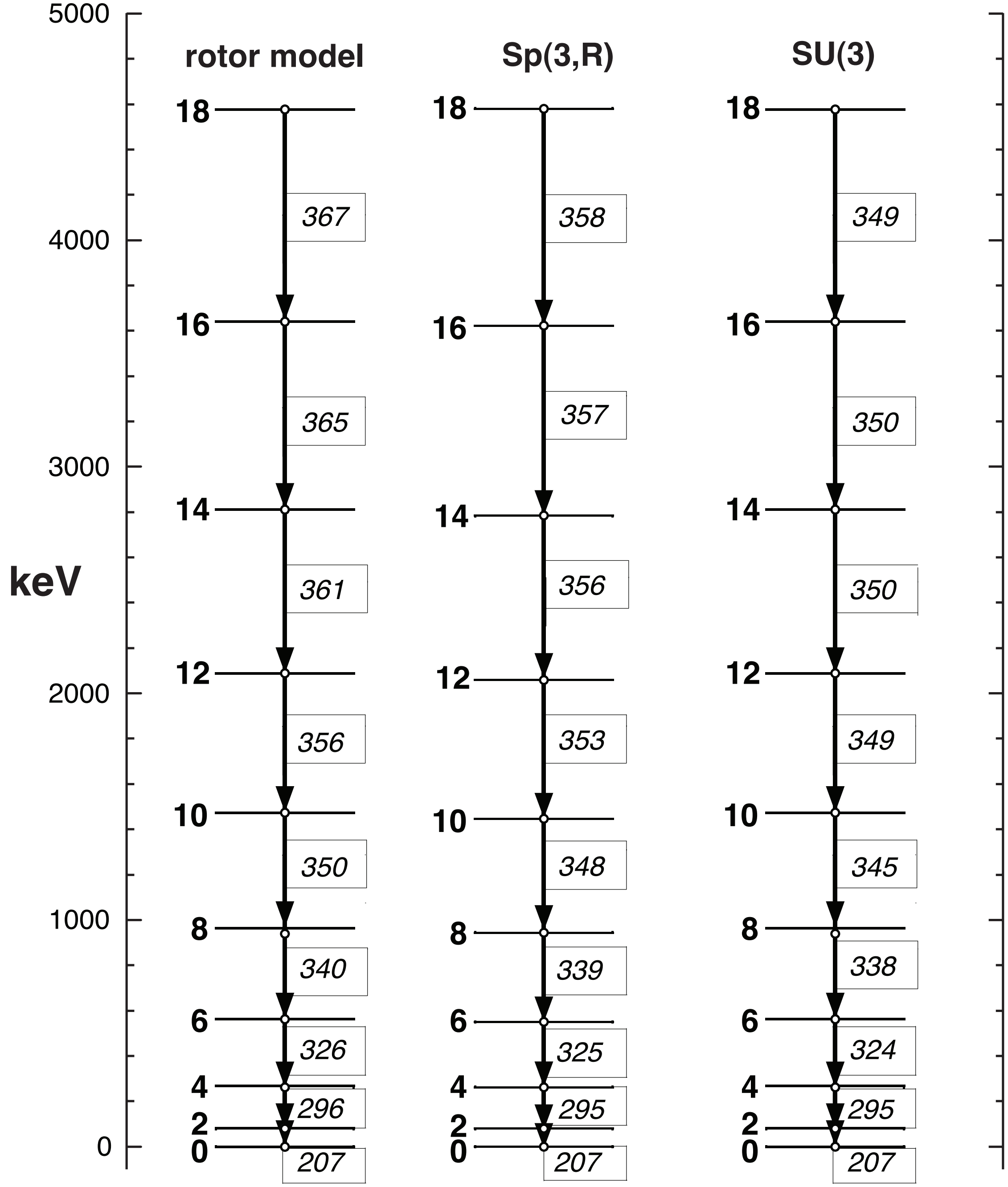
Jarrio, et al, Nucl. Phys. A528 (1991) 409

- **It might be sufficient to choose a small number Sp(3,R) irrep corresponding to the number of rotational bands to be described.**

Ground-state rotational band of ^{166}Er

Three fits to the same data:
(i)With a single Sp(3,R) irrep;
(ii)A phenomenological rigid rotor model;
(iii)A single SU(3) irrep.

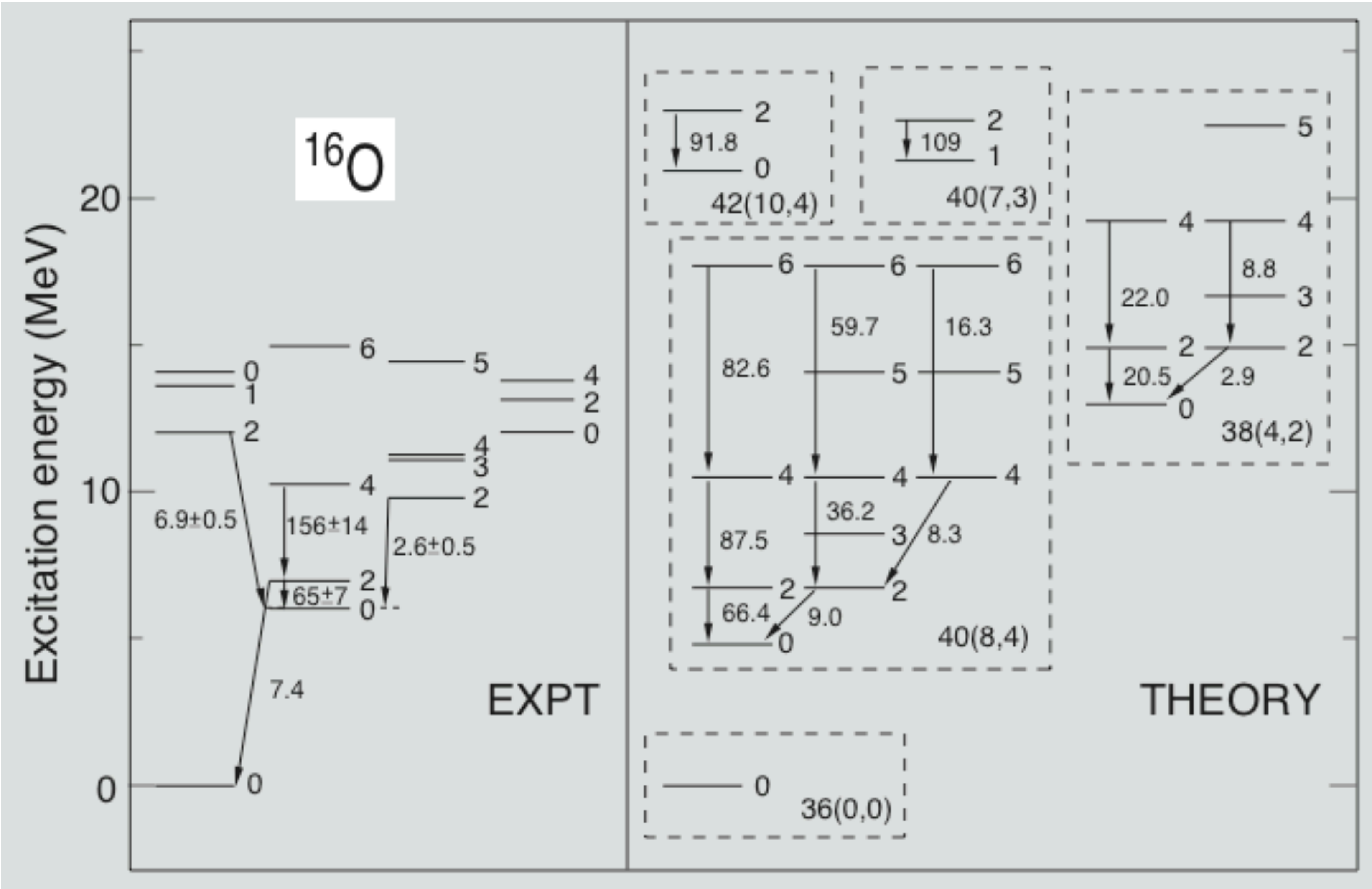
Note that in the microscopic Sp(3,R) model there is no need for an effective charge; observed E2 transition rates and moments of inertia are easily obtained



Bahri & Rowe
Nucl. Phys. A662, 125 (2000).

Shape coexistence is seen to occur in most (all?) nuclei

K.Heyde & J. Wood, Rev. Mod. Phys. (in press)



Conclusion

The path

$$\text{Phenomena} \longrightarrow \text{model} \longrightarrow \text{submodel of the SM}$$

leads to identification of the relevant truncated subspaces and coupling schemes for shell-model calculations;

e.g.

Spectra of single-closed shell nuclei indicate

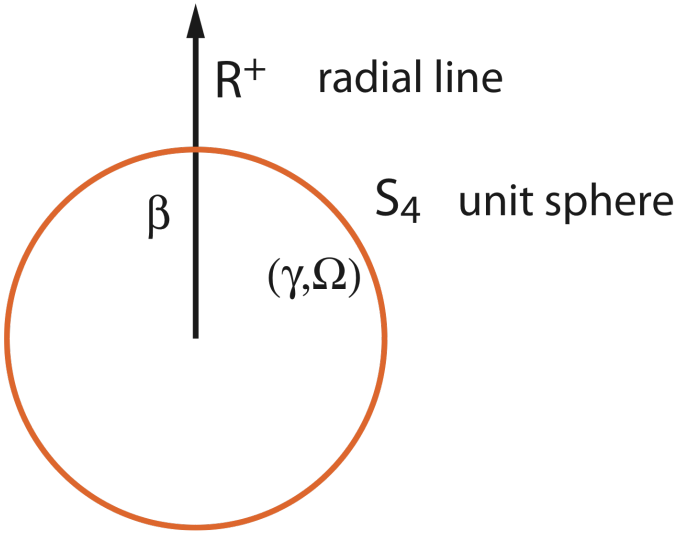
pair coupling in a spherical harmonic oscillator shell-model basis

whereas, deformation and rotational bands indicate

collective models and an Sp(3,R) > U(3) coupling scheme.

The shell model theory of effective interactions needs major revision to account for nuclear deformation and rotation.

Geometric structure



$$\alpha_\nu = \beta\, \mathcal{Y}_{12\nu}(\gamma,\Omega)$$

$$\mathcal{Y}_{12\nu}(\gamma,\Omega) \propto \cos\gamma\, \mathcal{D}^2_{0\nu}(\Omega) + \frac{1}{\sqrt{2}} \sin\gamma \left[\mathcal{D}^2_{2\nu}(\Omega) + \mathcal{D}^2_{-2,\nu}(\Omega)\right]$$

$$\Psi_{\lambda\mu;\nu LM} = \frac{1}{\beta^2} \mathcal{R}^\lambda_\mu(\beta)\, \mathcal{Y}_{\nu LM}(\gamma,\Omega)$$

$$\mathbb{R}^+ \quad \times \quad S_4$$

Algebraic structure

$$\mathrm{SU}(1,1) \quad \times \quad \mathrm{SO}(5) \supset \mathrm{SO}(3)$$

$$\mathcal{R}^\lambda_\mu(\beta) \quad \times \quad \mathcal{Y}_{\nu LM}(\gamma,\Omega)$$

From these structures we derive the basis wave functions for the radial wave functions and SO(5) spherical harmonics and matrix elements of all model observables of interest, including Clebsch-Gordan coefficients

The shell-model approach

The shell model is based on the premise that there is an ordered sequence of "harmonic oscillator" subspaces

$$\mathbb{H} = \mathbb{H}_1 \oplus \mathbb{H}_2 \oplus \mathbb{H}_3 \oplus \ldots$$

such that the corresponding expansion of low-lying states is a convergent sequence.

$$\Psi = \alpha_1\Psi_1 + \alpha_2\Psi_2 + \alpha_3\Psi_3 + \ldots$$

We need subspaces that factor into product of relative and centre-of-mass spaces and carry irreps of needed symmetry groups.

There are problems!

A schematic strategy for deriving effective interactions

- Choose a suitable harmonic-oscillator basis of single-particle states for the nucleus of interest.
- Solve the two-nucleon problem in this potential with realistic interactions and derive an effective two-nucleon on a large harmonic-oscillator subspace.
- Solve the three-nucleon problem on this subspace with the effective two-nucleon interactions and derive effective 2- and 3-nucleon effective interactions on a smaller harmonic-oscillator subspace.
- Continue this process for as far as needed.

If the computational facilities would enable such an algorithm to be completed without undue truncation at any step and if it were to converge rapidly in the sense that 4-nucleon and more many-nucleon interactions were small (or could be replaced by density-dependent 2-nucleon interactions) the strategy would have proved to be successful.

However, the result (even if it could be achieved) would probably be slowly convergent and (unnecessarily) complicated.

Shell-model coupling schemes

We need an optimally-ordered sequence of shell-model subspaces

$$\mathbb{H} = \mathbb{H}_1 \oplus \mathbb{H}_2 \oplus \mathbb{H}_3 \oplus \ldots$$

The subspaces should carry representations of suitable spectrum generating algebras to make the calculations feasible.

choose a suitable symmetry-based coupling scheme to give an ordering

There may be different optimal coupling schemes for different classes of nuclei: closed-shell, singly closed-shell, and doubly closed-shell nuclei.

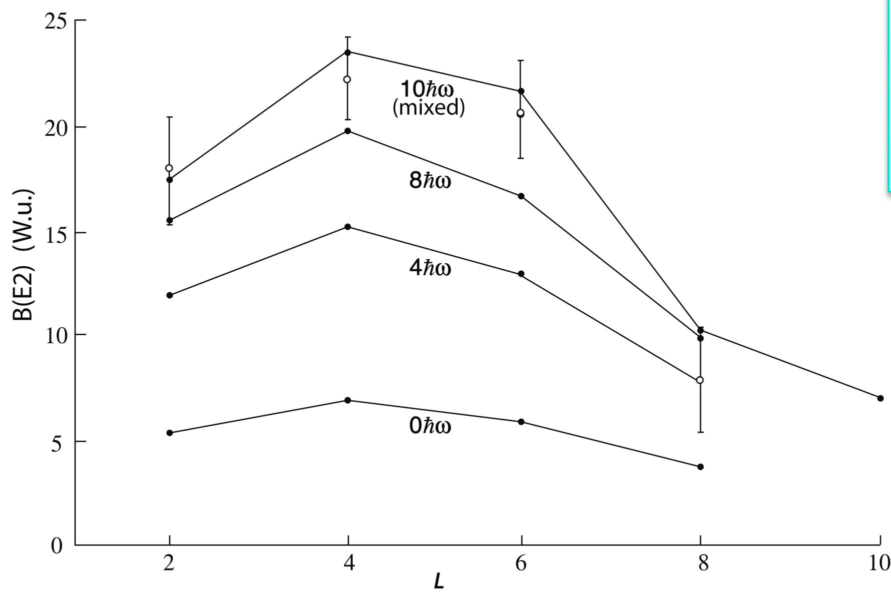
Operational sequence:

Data → model interpretation → algebraic model → shell model coupling scheme

Convergence in a harmonic-oscillator basis

For deformed nuclei, convergence of low-lying states of nuclei in a basis ordered by harmonic oscillator energies is very slow.

^{20}Ne



The dimensions of harmonic-oscillator shells increase almost exponentially with increasing oscillator quanta

J.P. Draayer, K.J. Weeks and G. Rosensteel, Nucl. Phys. A413 (1984) 215

G. Rosensteel and D.J. Rowe, Ann. Phys. 126 (1980) 343

Perturbation theory starting from the 0hw space is unlikely to work. So what do we do?

Coupling schemes suggested by models

The harmonic oscillator shell model:

- **Standard jj coupling: ordering of shells by number of quanta and spin-orbit interaction**

$$U(1)\times U(2j+1)\times SU_T(2)\supset SU_J(2)$$

- **J=0 pair-coupling model**

$$U(1)\times U(2j+1)\times SU_T(2) \ \supset Sp(2j+1)\supset SU_J(2)$$

seniority

- **L=0 pair-coupling model**

$$U(4)\times U(2l+1)\supset SU_S(2)\times SU_T(2) \ \times O(2l+1)\supset SU_J(2)$$

- **Collective model coupling scheme**

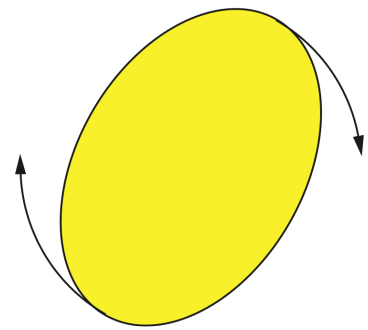
$$U(4)\times Sp(3,\mathbb{R})\supset SU_S(2)\times SU_T(2) \ \times U(3)\supset SU_J(2)$$

LSU-IOWA

The shell-model coupling scheme suggested by the Bohr collective model

The collective model:

Rotations and vibrations of a liquid drop



coordinates $\{\alpha_{\lambda\mu}\}$ given by surface shape parameter

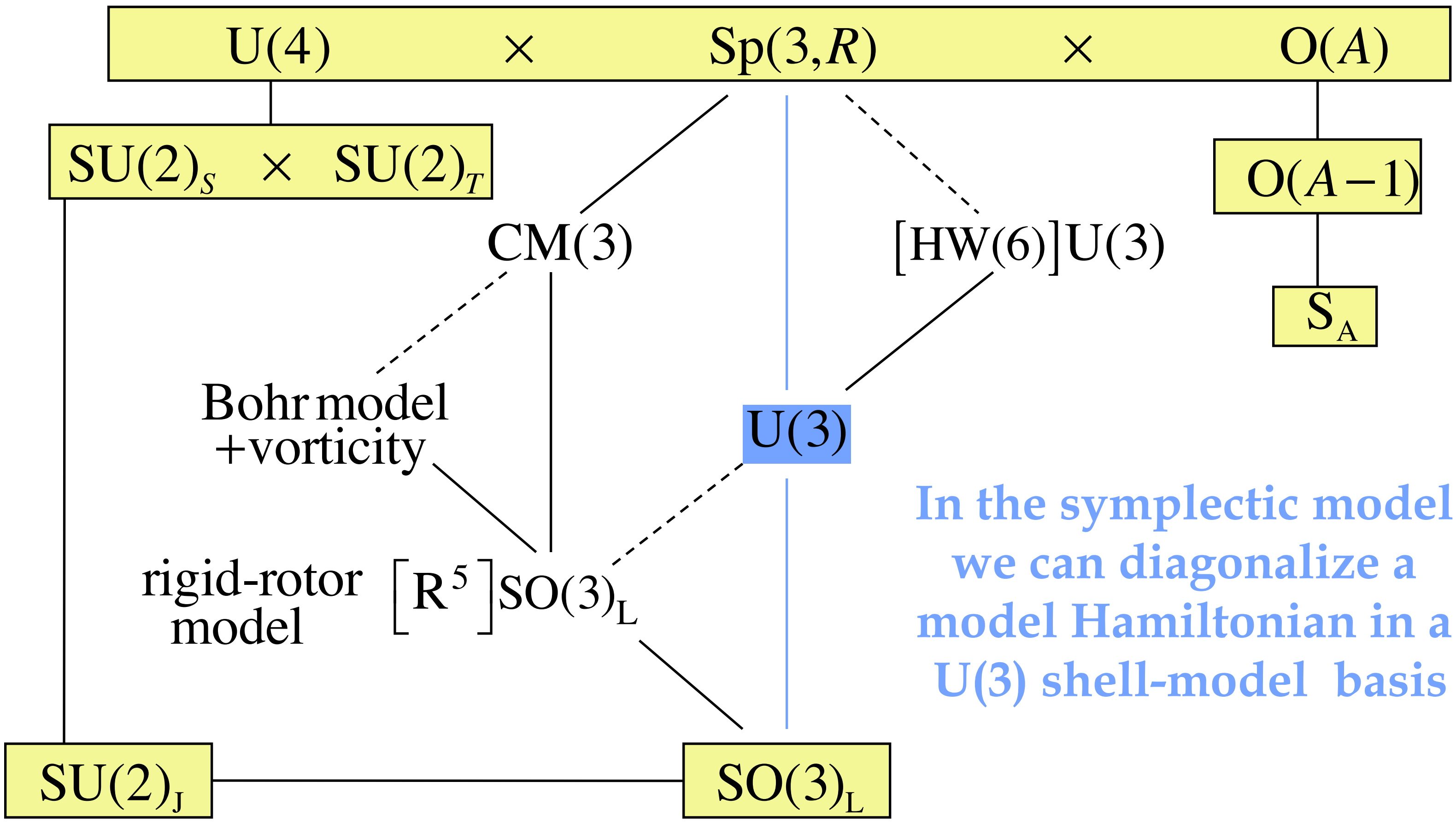
$$R(\vartheta,\varphi) = R_0\left[1+\sum_{\lambda\nu}\alpha_{\lambda\nu}Y^*_{\lambda\nu}(\vartheta,\varphi)\right]$$

classical variables $\{\alpha_{\lambda\nu},\dot{\alpha}_{\lambda\mu}\}$

quantum observables $\{\hat{\alpha}_{\lambda\nu},\hat{\pi}^{\lambda\mu}\}$ with $\hat{\pi}^{\lambda\mu} = -i\hbar\frac{\partial}{\partial\alpha_{\lambda\nu}}$

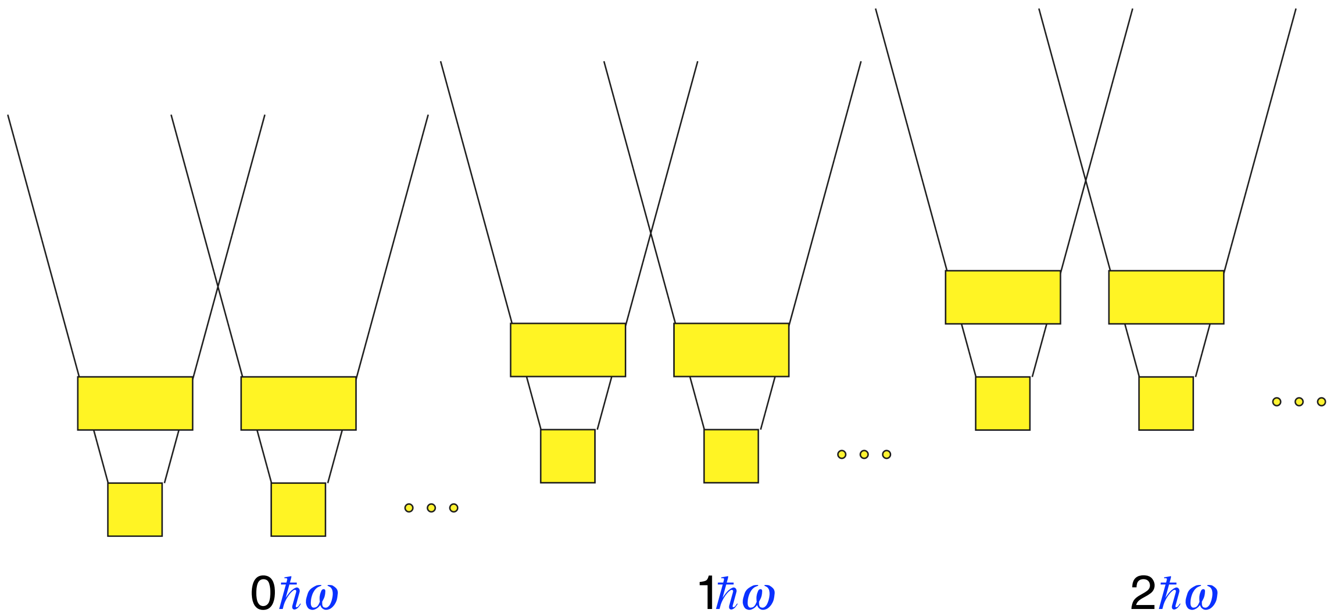
Dynamical groups of an LST coupling scheme

The symplectic shell model has a very rich algebraic structure



In the symplectic model we can diagonalize a model Hamiltonian in a U(3) shell-model basis

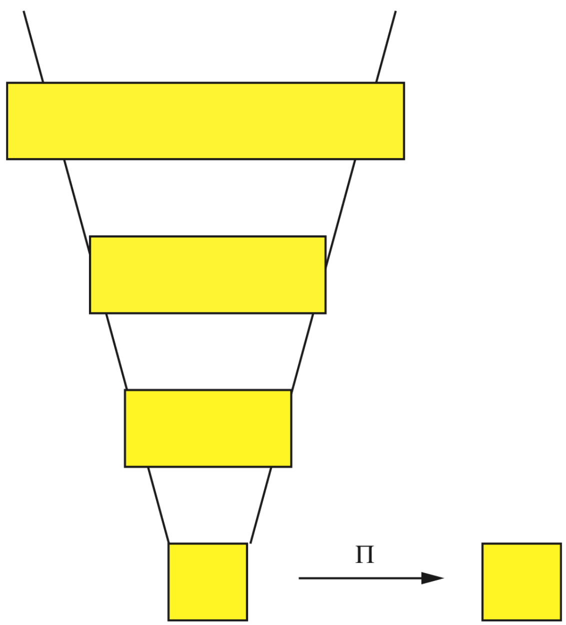
How do you order symplectic irreps?



(i) by preliminary calculations within each irrep;
(ii) by phenomenological calculations to fit the data,
(iii) by preliminary mean-field calculations, or …

Potential advantages of the U(4) x Sp(3,R) > U(3) coupling scheme

It is 'easy' to project a large Sp(3,ℝ)-coupled basis for a large space to a small U(3)-coupled effective shell-model and to map an algebraic sp(3,ℝ) Hamiltonian to an effective U(3) Hamiltonian.

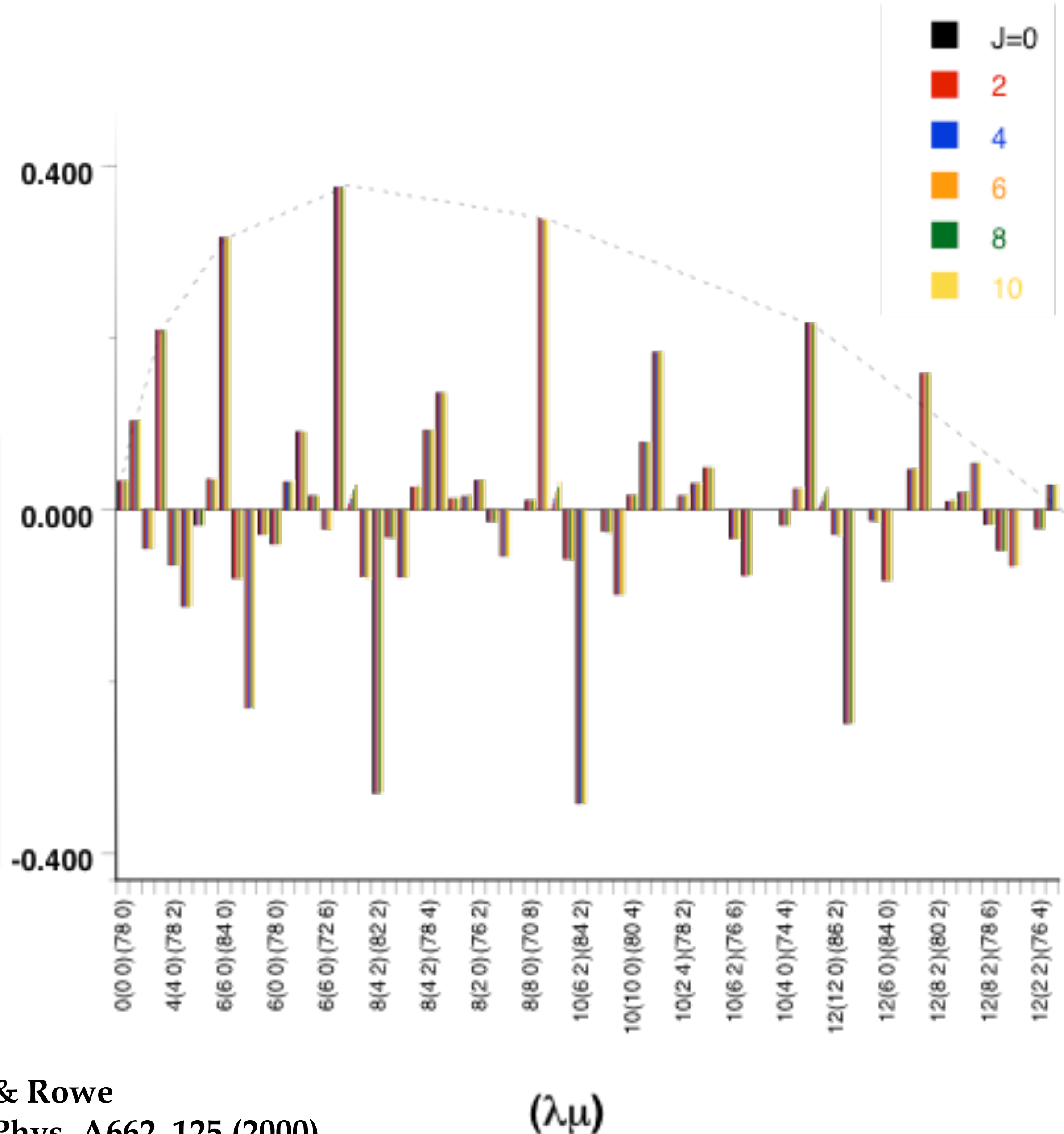


Each Sp(3,ℝ) irrep is simply replaced by its lowest-weight U(3) subspace

What about other SM operators and interactions?

Sp(3,ℝ) model wave functions for ^{166}Er expanded on an SU(3) basis

Because of the remarkable coherence (almost L independent coefs.), the huge linear combinations of U(3) irreps are well represented by an average (even lowest-weight) effective irrep (quasi-dynamical symmetry).



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